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Bianchi Type-I Space-Time with Variable Cosmological Constant

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Abstract A spatially homogeneous and anisotropic Bianchi type-I perfect fluid model is considered with variable cosmological constant. Einstein's field equations are solved by using a law of variation for mean Hubble's parameter, which is related to average scale factor and that yields a constant value of deceleration parameter. An exact and singular Bianchi-I model is presented, where the cosmological constant remains positive and decreases with the cosmic time. It is found that the solutions are consistent with the recent observations of type Ia supernovae. A detailed study of physical and kinematical properties of the model is carried out.

Keywords Bianchi models · Cosmology · Deceleration parameter · Cosmological constant

1 Introduction

There are many open questions in theoretical and experimental cosmologies, which are still under the consideration of cosmologists and need to be answered. One of the fundamental and challenging problems in cosmology is the cosmological constant (Λ) problem. Einstein considered the introduction of Λ-term into the field equations as a blunder of his life. But later on, researchers found that the Λ-term is very useful for describing many cosmological observations. Most of the astrophysical work continues to operate on the assumption $\Lambda = const$. For example, gravitational lens statistics [1], CMB anisotropies [2], CMB intensity [3], structure formation [4, 5], cluster abundance [6], etc. At the same time the studies of Barr [7], Peebles and Ratra [8], Moffat [9], Frieman et al. [10], Abramo et al. [11] and many others have shown to give rise an effective cosmological term that decays with time. The discussions of Ratra and Peebles [12], Dolgov [13–15] and Sahni and Starobinsky [16] on the cosmological constant problem and on cosmology with a time-varying cosmological constant reveal that in the absence of any interaction with matter or radiation, a solution of

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Einstein's equations and the assumed equation of covariant conservation of stress-energy with a time-varying Λ can be found. For these solutions the conservation of energy requires decrease in the energy density of matter or radiation.

Recent cosmological observations by High-z Supernovae Team and Supernovae Cosmological Project (Garnavich et al. [17, 18], Perlmutter et al. [19–21], Riess et al. [22, 23], Schmidt et al. [24]) strongly favor a significant and positive Λ with magnitude $\Lambda(Gh/c^3) \approx 10^{-123}$. It has been a consequence of the study of more than 50 type Ia supernovae with redshifts in the range $0.10 \le z \le 0.83$ and it predicts Friedmann models with negative pressure matter such as a cosmological constant, domain walls or cosmic strings [25–27]. Thus the weight of observational evidence is now converging on a non-zero value of the cosmological term Λ. At the same time theorists are increasingly exploring the possibility that this parameter is a dynamical one whose effective value in the early universe may have been quite different from the one we measure today.

The phenomenological Λ-decay scenarios have been considered by a number of authors [28–35]. Of the special interest is the assumption $\Lambda \propto a^{-2}$ (a is the scale factor of the Robertson-Walker metric) by Chen and Wu [30], which has been considered/modified by several authors (Abdel-Rahaman [36, 37], Carvalho et al. [38], Waga [39], Silveira and Waga [40], Overduin [41] and Vishwakarma [42]). In particular, Overduin [41] considered the assumptions $\Lambda \propto a^{-n}$ and $\Lambda \propto H^2$ in FRW models and showed their compatibility with various astronomical observations. Singh et al. [43] have considered these phenomenological assumptions in Bianchi type-I models. Berman [44] has also used the relation $\Lambda \propto H^2$ in the investigation of inflationary Brans-Dicke cosmology.

A number of authors have instead argued in favor of the dependence $\Lambda \propto t^{-2}$. Berman and Som [45] pointed out that the relation $\Lambda \propto t^{-2}$ seems to play a major role in cosmology. In fact, Berman et al. [46] found this relation in Brans-Dicke static models. Berman [47] predicted it in a static universe with Endo-Fukui modified Brans-Dicke cosmology. Berman and Som [45] and Berman [48] investigated it again in general Brans-Dicke models which obey the perfect gas equation of state. The same relation was obtained by Bertolami [49, 50], Beesham [51, 52], Singh et al. [53]. Pradhan and Kumar [54] found this relation while investigating locally rotationally symmetric (LRS) Bianchi type-I models with time-dependent cosmological term Λ. One may also refer to Saha [55–57], Peebles and Ratra [58], Padmanabhan [59], Pradhan et al. [60–64] and Chakravarty and Biswas [65] for studying the role of cosmological constant in the evolution of universe.

In this paper our intention is to obtain a physically realistic and exact Bianchi type-I model with a time-varying Λ-term. We also intend to explore the possibilities of Λ-decay scenarios as discussed above. Therefore, in what follows, a spatially homogeneous and anisotropic Bianchi type-I space-time filled with perfect fluid is considered, where the Λ-term is allowed to vary with time. Exact solutions of Einstein's field equations are found by using variation for mean Hubble's parameter that is related to average scale factor and yields a constant value of deceleration parameter (DP). Physical behavior of the model is discussed in detail followed by concluding remarks.

2 Model and Field Equations

The spatially homogeneous and anisotropic Bianchi-I space-time is described by the line element

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2, \tag{1}$$

where $A(t)$, $B(t)$ and $C(t)$ are the metric functions of cosmic time t.

We define $a = (ABC)^{\frac{1}{3}}$ as the average scale factor so that the Hubble's parameter in anisotropic models may be defined as

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right), \tag{2}$$

where an over dot denotes derivative with respect to the cosmic time t.

Also we have

$$H = \frac{1}{3}(H_1 + H_2 + H_3), \tag{3}$$

where $H_1 = \frac{\dot{A}}{A}$, $H_2 = \frac{\dot{B}}{B}$ and $H_3 = \frac{\dot{C}}{C}$ are directional Hubble's factors in the directions of x, y and z respectively.

In general relativity, the Bianchi identities for the Einstein tensor G_{ij} and the vanishing covariant divergence of the energy momentum tensor T_{ij} together with imply that the cosmological term Λ is constant. In theories with a variable Λ-term, one either introduces new terms (involving scalar fields, for instance) into the left hand side of the Einstein's field equations to cancel the non-zero divergence of Λg_{ij} [66, 67] or interprets Λ as a matter source and moves it to the right hand side of the field equations [68], in which case energy momentum conservation is understood to mean $T^{*ij}_{;j} = 0$, where $T^{*}_{ij} = T_{ij} - (\Lambda/8\pi G)g_{ij}$. The two approaches are of course equivalent for a given theory [41]. Here we follow the later approach and assume the perfect fluid energy momentum tensor augmented with the Λ-term as

$$T^{*}_{ij} = (\rho + p)u_i u_j + \left(p - \frac{\Lambda}{8\pi G}\right) g_{ij} \tag{4}$$

together with a perfect gas equation of state

$$p = \gamma\rho, \quad 0 \le \gamma \le 1 \tag{5}$$

where ρ and p are, respectively the energy density and pressure of the cosmic fluid and u_i is the four velocity such that $g_{ij}u_i u_j = -1$.

For Bianchi type-I space-time (1) and the energy momentum tensor (4), the Einstein's field equations

$$G_{ij} = -8\pi G T^{*}_{ij} \tag{6}$$

yield the following four independent equations:

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -8\pi G p + \Lambda, \tag{7}$$

$$\frac{\ddot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{C}\dot{A}}{CA} = -8\pi G p + \Lambda, \tag{8}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -8\pi G p + \Lambda, \tag{9}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} = 8\pi G\rho + \Lambda. \tag{10}$$

The energy conservation equation

$$T^{*ij}_{;j} = 0, \tag{11}$$

which is a consequence of the field equations (7)–(10), leads to

$$\dot{\rho} + (\rho + p)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{\dot{\Lambda}}{8\pi G} = 0. \tag{12}$$

The field equations (7)–(10) are four equations involving six unknowns A, B, C, p, ρ and Λ. So for complete determinacy of the system together with (5), we need one more relation among the variables that we shall obtain in the following section by applying a law of variation for Hubble's parameter recently used by the present authors [69] and solve the field equations.

3 Solution of the Field Equations

In order to solve Einstein's field equations, we normally assume a form for the matter content or suppose that the space-time admits killing vector symmetries. Solutions to the field equations may also be generated by applying a law of variation for Hubble's parameter, which was first proposed by Berman [70] in FRW models and that yields a constant value of DP. Later on, several authors have considered FRW and Bianchi type cosmological models in general relativity and scalar-tensor theories with constant DP (see [69] and references therein). Recently Reddy et al. have presented LRS Bianchi type-I models with constant DP in scalar-tensor [71], scale-covariant [72] and Brans-Bicke [73] theories of gravitation. The present authors [69, 74–78] have investigated Bianchi type-I and II cosmological models with constant DP in general relativity and some scalar-tensor theories by using a law of variation for mean Hubble's parameter. According to the law, the mean Hubble's parameter is related to the average scale factor as

$$H = Da^{-n}, \tag{13}$$

where D and n are positive constants.

The DP q is defined by

$$q = -\frac{a\ddot{a}}{\dot{a}^2}. \tag{14}$$

From (2) and (13), we get

$$\frac{\dot{a}}{a} = Da^{-n}, \tag{15}$$

which on integration leads to

$$a = (nDt + C_1)^{\frac{1}{n}}, \tag{16}$$

where C_1 is a constant of integration.

Substituting (16) into (14), we get

$$q = n - 1. \tag{17}$$

This shows that the law (13) leads to a constant value of DP.

From (7)–(9), the metric functions can be obtained as [69]

$$A(t) = a_1 a \exp\left(b_1 \int a^{-3} dt\right), \tag{18}$$

$$B(t) = a_2 a \exp\left(b_2 \int a^{-3} dt\right), \tag{19}$$

$$C(t) = a_3 a \exp\left(b_3 \int a^{-3} dt\right), \tag{20}$$

where a_1, b_1, a_2, b_2, a_3 and b_3 are constants, which satisfy the following two relations:

$$a_1 a_2 a_3 = 1, \qquad b_1 + b_2 + b_3 = 0. \tag{21}$$

Using (16) into (18)–(20), we get the following expressions for scale factors:

$$A(t) = a_1 (nDt + C_1)^{\frac{1}{n}} \exp\left[\frac{b_1}{D(n-3)}(nDt + C_1)^{\frac{n-3}{n}}\right], \tag{22}$$

$$B(t) = a_2 (nDt + C_1)^{\frac{1}{n}} \exp\left[\frac{b_2}{D(n-3)}(nDt + C_1)^{\frac{n-3}{n}}\right], \tag{23}$$

$$C(t) = a_3 (nDt + C_1)^{\frac{1}{n}} \exp\left[\frac{b_3}{D(n-3)}(nDt + C_1)^{\frac{n-3}{n}}\right]. \tag{24}$$

Substituting (22)–(24) into (9) and (10), and solving with (5), we get the pressure, energy density and Λ-term of the model as

$$p = \frac{\gamma}{4\pi G(1+\gamma)}\left[nD^2(nDt + C_1)^{-2} - (b_1^2 + b_2^2 + b_1 b_2)(nDt + C_1)^{\frac{-6}{n}}\right], \tag{25}$$

$$\rho = \frac{1}{4\pi G(1+\gamma)}\left[nD^2(nDt + C_1)^{-2} - (b_1^2 + b_2^2 + b_1 b_2)(nDt + C_1)^{\frac{-6}{n}}\right], \tag{26}$$

$$\Lambda = \frac{D^2(3 - 2n + 3\gamma)}{(1+\gamma)}(nDt + C_1)^{-2} + \frac{(1-\gamma)}{(1+\gamma)}(b_1^2 + b_2^2 + b_1 b_2)(nDt + C_1)^{\frac{-6}{n}}. \tag{27}$$

In view of (21), one may observe that the solutions (22)–(27) satisfy the energy conservation equation (12) identically and hence represent exact solutions of the Einstein's field equations (7)–(10). Using (22)–(24) into (1), we get the following exact Bianchi type-I model:

$$ds^2 = -dt^2 + T^{\frac{2}{n}}\left(a_1^2 e^{kb_1 T^{\frac{n-3}{n}}} dx^2 + a_2^2 e^{kb_2 T^{\frac{n-3}{n}}} dy^2 + a_3^2 e^{kb_3 T^{\frac{n-3}{n}}} dz^2\right), \tag{28}$$

where $T = nDt + C_1$ and $k = \frac{2}{D(n-3)}$.

The model (28) being totally anisotropic, is more general than the model investigated by Pradhan and Kumar [54] who have presented LRS Bianchi type-I model with variable Λ. More precisely if we estimate the exponential factor in (22) by first two terms of its expansion and the exponential factors in (23) and (24) by unity each, then the above solutions reduce to the solutions obtained by Pradhan and Kumar [54] with some proper choice of constants such as $a_2 = a_3 = 1$, $n = \frac{2}{1-m}$ etc. Thus we have generalized the work of Pradhan and Kumar [54]. It may be noted that the work of Pradhan and Kumar [54] is itself a generalization of some important works by Majumder [79], Hajj-Boutros and Sfeila [80] and Shri Ram [81].

Now we find the expressions for some other cosmological parameters of the model. The anisotropy parameter A is defined as

$$\mathrm{A} = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2. \tag{29}$$

The directional Hubble factors H_i $(i = 1, 2, 3)$ as defined in (3) are given by

$$H_i = D(nDt + C_1)^{-1} + b_i(nDt + C_1)^{\frac{-3}{n}}. \tag{30}$$

The expansion scalar is given by

$$\Theta = 3H = 3D(nDt + C_1)^{-1}. \tag{31}$$

Using (30) and (31) into (29), we get

$$\mathrm{A} = \frac{1}{3D^2}(b_1^2 + b_2^2 + b_3^2)(nDt + C_1)^{\frac{2n-6}{n}}. \tag{32}$$

The volume and shear scalar of the model are given by

$$V^3 = (nDt + C_1)^{\frac{3}{n}}, \tag{33}$$

$$\sigma^2 = \frac{1}{3}\left[(b_1 - b_2)^2 + (b_2 - b_3)^2 + (b_3 - b_1)^2\right](nDt + C_1)^{\frac{-6}{n}}. \tag{34}$$

4 Physical Behavior of the Model

It is observed that the spatial volume is zero at $t = t_0$ where $t_0 = -C_1/nD$ and expansion scalar is infinite, which shows that the universe starts evolving with zero volume at $t = t_0$ with an infinite rate of expansion. The scale factors also vanish at $t = t_0$ and hence the model has a point singularity at the initial epoch. The pressure, energy density, cosmological term, Hubble's parameter and shear scalar diverge at the initial singularity. The anisotropy parameter also tends to infinity at the initial epoch provided $n < 3$. The universe exhibits the power-law expansion after the big bang impulse. As t increases the scale factors and spatial volume increase but the expansion scalar decreases. Thus the rate of expansion slows down with increase in time. Also ρ, p, H_1, H_2, H_3, A and σ^2 decrease as t increases.

We observe that the Λ-term has an effective positive value that decreases with time i.e. the Λ-term decays as the cosmic evolution progresses. This is consistent with the predictions of several authors [7–11] and also with the recent cosmological observations [17–27] as discussed in the introduction. Further if we assume $b_1 = b_2 = C_1 = 0$, then the cosmological term varies as the inverse square of time i.e. $\Lambda \propto t^{-2}$. This form of Λ has been obtained by several authors [45–54]. Moreover in view of (31), we find that $\Lambda \propto H^2$, which has again been considered by many authors [35, 41, 43, 44].

As $t \to \infty$, scale factors and volume become infinite whereas ρ, p, Λ, H_1, H_2, H_3, Θ, A and σ^2 tend to zero. Therefore the model would essentially give an empty universe for large time t. The ratio σ/Θ tends to zero as $t \to \infty$ provided $n < 3$. So the model approaches isotropy for large values of t. Thus the model represents shearing, non-rotating and expanding model of the universe with a big bang start approaching to isotropy at late times. The integral

$$\int_{t_0}^{t} \frac{dt'}{V(t')} = \frac{1}{D(n-1)} \left[(nDt' + C_1)^{\frac{n-1}{n}} \right]_{t_0}^{t}$$

is finite provided $n \neq 1$. Therefore a particle horizon exists in this model.

Further it is found that the above solutions are not valid for $n = 3$. For $n = 3$, the spatial volume grows linearly with cosmic time. For $n > 1$, $q > 0$; therefore the model represents a decelerating universe. For $n \leq 1$, we get $-1 < q \leq 0$, which implies an accelerating model of the universe. Also recent observations of type Ia supernovae (Perlmutter et al. [19–21], Riess et al. [22, 23], Tonry et al. [82], Knop et al. [83] and John [84]) reveal that the present universe is accelerating and value of DP lies somewhere in the range $-1 < q \leq 0$. It follows that the solutions obtained in this model are consistent with the observations. Clearly the first term inside the square brackets of (25) and (26) is dominant over the second one. Therefore the energy conditions given by Ellis [85]

(i) $(\rho + p) > 0$,
(ii) $(\rho + 3p) > 0$,
(iii) $\rho > 0$,

and the dominant energy conditions given by Hawking and Ellis [86]

(i) $(\rho - p) \geq 0$,
(ii) $(\rho + p) \geq 0$,

are satisfied trivially.

5 Conclusion

In this paper we have presented exact solutions of a spatially homogeneous and totally anisotropic Bianchi-I space-time filled with perfect fluid and time-dependent Λ-term by using a law of variation for Hubble's parameter that yields a constant value of DP. The solutions generalize the work of Pradhan and Kumar [54] and lead to an exact and physically realistic model of the universe. In this model all the matter and radiation is concentrated at the big bang epoch and the cosmic expansion is driven by the big bang impulse. The model has a point singularity at the initial epoch as the scale factors and volume vanish at this moment. The universe has singular origin and it exhibits power-law expansion after the big bang impulse. The Λ-term is found to have an effective positive value, which decreases with the cosmic time. Also the form of the Λ-term obtained in this model is found to be more general than the forms already investigated by several authors [45–54]. The model represents shearing, non-rotating and expanding universe, which approaches to isotropy for large values of t. Therefore the anisotropy damps out during the cosmic evolution. The solutions obtained in the model are consistent with the recent observations [19–23, 82–84] of type Ia supernovae as discussed in the physical behavior of the model. Finally, it is possible that the phenomenological term-Λ has played an important role in the evolution of our dynamic universe and will continue to do so. The solutions presented in this paper are new and may be useful for better understanding of the evolution of universe in anisotropic Bianchi-I space-time with time dependent Λ-term in general relativity.

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